Many algebraic expressions that describe costs of environmental projects are examples of *rational expressions*. First we will define rational expressions. Then we will review how to perform operations with such expressions.

1 Specify numbers that must be excluded from the domain of a rational expression.

Rational Expressions

A **rational expression** is the quotient of two polynomials. Some examples are

$$\frac{x-2}{4}, \quad \frac{4}{x-2}, \quad \frac{x}{x^2-1}, \quad \text{and} \quad \frac{x^2+1}{x^2+2x-3}.$$

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Because rational expressions indicate division and division by zero is undefined, **we must exclude numbers from a rational expression's domain that make the denominator zero.**

EXAMPLE 1 Excluding Numbers from the Domain

Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{4}{x-2}$ **b.** $\frac{x}{x^2-1}$.

Solution To determine the numbers that must be excluded from each domain, examine the denominators.

a. $\frac{4}{x-2}$ **b.** $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$

This denominator would equal zero if $x = 2$.

This factor would equal zero if $x = -1$.

This factor would equal zero if $x = 1$.

For the rational expression in part (a), we must exclude 2 from the domain. For the rational expression in part (b), we must exclude both -1 and 1 from the domain. These excluded numbers are often written to the right of a rational expression:

$$\frac{4}{x-2}, x \neq 2 \qquad \frac{x}{x^2-1}, x \neq -1, x \neq 1$$

Check Point 1 Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{7}{x+5}$ **b.** $\frac{x}{x^2-36}$.

2 Simplify rational expressions.

Simplifying Rational Expressions

A rational expression is **simplified** if its numerator and denominator have no common factors other than 1 or -1. The following procedure can be used to simplify rational expressions:

Simplifying Rational Expressions

1. Factor the numerator and the denominator completely.
2. Divide both the numerator and the denominator by any common factors.

EXAMPLE 2 **Simplifying Rational Expressions**

Simplify:

a. $\frac{x^3 + x^2}{x + 1}$ **b.** $\frac{x^2 + 6x + 5}{x^2 - 25}$.

Solution

a. $\frac{x^3 + x^2}{x + 1} = \frac{x^2(x + 1)}{x + 1}$ Factor the numerator. Because the denominator is $x + 1$, $x \neq -1$.

$= \frac{x^2\cancel{(x + 1)}^{1}}{\cancel{x + 1}_{1}}$ Divide out the common factor, $x + 1$.

$= x^2, x \neq -1$ Denominators of 1 need not be written because $\frac{a}{1} = a$.

b. $\frac{x^2 + 6x + 5}{x^2 - 25} = \frac{(x + 5)(x + 1)}{(x + 5)(x - 5)}$ Factor the numerator and denominator. Because the denominator is $(x + 5)(x - 5)$, $x \neq -5$ and $x \neq 5$.

$= \frac{\cancel{(x + 5)}^{1}(x + 1)}{\cancel{(x + 5)}_{1}(x - 5)}$ Divide out the common factor, $x + 5$.

$= \frac{x + 1}{x - 5}, \quad x \neq -5, \quad x \neq 5$

Check Point 2 Simplify:

a. $\frac{x^3 + 3x^2}{x + 3}$ **b.** $\frac{x^2 - 1}{x^2 + 2x + 1}$.

3 Multiply rational expressions.

Multiplying Rational Expressions

The product of two rational expressions is the product of their numerators divided by the product of their denominators. Here is a step-by-step procedure for multiplying rational expressions:

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.

EXAMPLE 3 Multiplying Rational Expressions

Multiply: $\dfrac{x-7}{x-1} \cdot \dfrac{x^2-1}{3x-21}$.

Solution

$$\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$$

This is the given multiplication problem.

$$= \frac{x-7}{x-1} \cdot \frac{(x+1)(x-1)}{3(x-7)}$$

Factor as many numerators and denominators as possible. Because the denominators have factors of $x-1$ and $x-7$, $x \neq 1$ and $x \neq 7$.

$$= \frac{\overset{1}{\cancel{x-7}}}{\underset{1}{\cancel{x-1}}} \cdot \frac{(x+1)\overset{1}{\cancel{(x-1)}}}{3\underset{1}{\cancel{(x-7)}}}$$

Divide numerators and denominators by common factors.

$$= \frac{x+1}{3}, x \neq 1, x \neq 7$$

Multiply the remaining factors in the numerators and denominators.

These excluded numbers from the domain must also be excluded from the simplified expression's domain.

Check Point 3 Multiply:

$$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9}.$$

4 Divide rational expressions.

Dividing Rational Expressions

The quotient of two rational expressions is the product of the first expression and the multiplicative inverse, or reciprocal, of the second expression. The reciprocal is found by interchanging the numerator and the denominator. Thus, **we find the quotient of two rational expressions by inverting the divisor and multiplying.**

EXAMPLE 4 Dividing Rational Expressions

Divide: $\dfrac{x^2-2x-8}{x^2-9} \div \dfrac{x-4}{x+3}$.

Solution

$$\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$$

This is the given division problem.

$$= \frac{x^2-2x-8}{x^2-9} \cdot \frac{x+3}{x-4}$$

Invert the divisor and multiply.

$$= \frac{(x-4)(x+2)}{(x+3)(x-3)} \cdot \frac{x+3}{x-4}$$

Factor as many numerators and denominators as possible. For nonzero denominators, $x \neq -3$, $x \neq 3$, and $x \neq 4$.

$$= \frac{\overset{1}{\cancel{(x-4)}}(x+2)}{\underset{1}{\cancel{(x+3)}}(x-3)} \cdot \frac{\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-4)}}}$$

Divide numerators and denominators by common factors.

$$= \frac{x+2}{x-3}, x \neq -3, x \neq 3, x \neq 4$$

Multiply the remaining factors in the numerators and in the denominators.

Check Point 4 Divide:

$$\frac{x^2 - 2x + 1}{x^3 + x} \div \frac{x^2 + x - 2}{3x^2 + 3}.$$

5 Add and subtract rational expressions.

Adding and Subtracting Rational Expressions with the Same Denominator

We add or subtract rational expressions with the same denominator by (1) adding or subtracting the numerators, (2) placing this result over the common denominator, and (3) simplifying, if possible.

EXAMPLE 5 **Subtracting Rational Expressions with the Same Denominator**

Subtract: $\dfrac{5x + 1}{x^2 - 9} - \dfrac{4x - 2}{x^2 - 9}.$

Solution

Don't forget the parentheses.

$$\frac{5x + 1}{x^2 - 9} - \frac{4x - 2}{x^2 - 9} = \frac{5x + 1 - (4x - 2)}{x^2 - 9}$$

Subtract numerators and include parentheses to indicate that both terms are subtracted. Place this difference over the common denominator.

$$= \frac{5x + 1 - 4x + 2}{x^2 - 9}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x + 3}{x^2 - 9}$$

Combine like terms.

$$= \frac{\overset{1}{\cancel{x + 3}}}{\underset{1}{\cancel{(x + 3)}}(x - 3)}$$

Factor and simplify ($x \neq -3$ and $x \neq 3$).

$$= \frac{1}{x - 3}, \; x \neq -3, x \neq 3$$

Study Tip

Example 5 shows that when a numerator is being subtracted, we must subtract every term in that expression.

Check Point 5 Subtract: $\dfrac{x}{x + 1} - \dfrac{3x + 2}{x + 1}.$

Adding and Subtracting Rational Expressions with Different Denominators

Rational expressions that have no common factors in their denominators can be added or subtracted using one of the following properties:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, \; b \neq 0, d \neq 0.$$

The denominator, bd, is the product of the factors in the two denominators. Because we are considering rational expressions that have no common factors in their denominators, the product bd gives the least common denominator.

EXAMPLE 6 **Subtracting Rational Expressions Having No Common Factors in Their Denominators**

Subtract: $\frac{x+2}{2x-3} - \frac{4}{x+3}$.

Solution We need to find the least common denominator. This is the product of the distinct factors in each denominator, namely $(2x-3)(x+3)$. We can therefore use the subtraction property given previously as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

$$\frac{x+2}{2x-3} - \frac{4}{x+3} = \frac{(x+2)(x+3)-(2x-3)4}{(2x-3)(x+3)}$$

Observe that $a = x+2$, $b = 2x-3$, $c = 4$, and $d = x+3$.

$$= \frac{x^2+5x+6-(8x-12)}{(2x-3)(x+3)}$$

Multiply.

$$= \frac{x^2+5x+6-8x+12}{(2x-3)(x+3)}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x^2-3x+18}{(2x-3)(x+3)}, x \neq \frac{3}{2}, x \neq -3$$

Combine like terms in the numerator.

Check Point 6 Add: $\frac{3}{x+1} + \frac{5}{x-1}$.

The **least common denominator**, or LCD, of several rational expressions is a polynomial consisting of the product of all prime factors in the denominators, with each factor raised to the greatest power of its occurrence in any denominator. When adding and subtracting rational expressions that have different denominators with one or more common factors in the denominators, it is efficient to find the least common denominator first.

Finding the Least Common Denominator

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of the factors from the list in step 3. This product is the least common denominator.

EXAMPLE 7 **Finding the Least Common Denominator**

Find the least common denominator of

$$\frac{7}{5x^2+15x} \quad \text{and} \quad \frac{9}{x^2+6x+9}.$$

Solution

Step 1 Factor each denominator completely.

$$5x^2 + 15x = 5x(x + 3)$$
$$x^2 + 6x + 9 = (x + 3)^2 \quad \text{or} \quad (x + 3)(x + 3)$$

Factors are 5, x, and $x + 3$. $\quad \dfrac{7}{5x^2 + 15x} \qquad \dfrac{9}{x^2 + 6x + 9}$ Factors are $x + 3$ and $x + 3$.

Step 2 List the factors of the first denominator.

$$5, x, x + 3$$

Step 3 Add any unlisted factors from the second denominator. One factor of $x^2 + 6x + 9$ is already in our list. That factor is $x + 3$. However, the other factor of $x + 3$ is not listed in step 2. We add a second factor of $x + 3$ to the list. We have

$$5, x, x + 3, x + 3.$$

Step 4 The least common denominator is the product of all factors in the final list. Thus,

$$5x(x + 3)(x + 3) \quad \text{or} \quad 5x(x + 3)^2$$

is the least common denominator.

Check Point 7 Find the least common denominator of

$$\frac{3}{x^2 - 6x + 9} \quad \text{and} \quad \frac{7}{x^2 - 9}.$$

Finding the least common denominator for two (or more) rational expressions is the first step needed to add or subtract the expressions.

Adding and Subtracting Rational Expressions That Have Different Denominators

1. Find the LCD of the rational expressions.
2. Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do so, multiply the numerator and the denominator of each rational expression by any factor(s) needed to convert the denominator into the LCD.
3. Add or subtract numerators, placing the resulting expression over the LCD.
4. If possible, simplify the resulting rational expression.

EXAMPLE 8 Adding Rational Expressions with Different Denominators

Add: $\dfrac{x + 3}{x^2 + x - 2} + \dfrac{2}{x^2 - 1}$.

Solution

Step 1 Find the least common denominator. Start by factoring the denominators.

$$x^2 + x - 2 = (x + 2)(x - 1)$$
$$x^2 - 1 = (x + 1)(x - 1)$$

The factors of the first denominator are $x + 2$ and $x - 1$. The only factor from the second denominator that is not listed is $x + 1$. Thus, the least common denominator is

$$(x + 2)(x - 1)(x + 1).$$

Step 2 Write equivalent expressions with the LCD as denominators. We must rewrite each rational expression with a denominator of $(x+2)(x-1)(x+1)$. We do so by multiplying both the numerator and the denominator of each rational expression by any factor(s) needed to convert the expression's denominator into the LCD.

$$\frac{x+3}{(x+2)(x-1)}\cdot\frac{x+1}{x+1}=\frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)} \qquad \frac{2}{(x+1)(x-1)}\cdot\frac{x+2}{x+2}=\frac{2(x+2)}{(x+2)(x-1)(x+1)}$$

Multiply the numerator and denominator by $x+1$ to get $(x+2)(x-1)(x+1)$, the LCD.

Multiply the numerator and denominator by $x+2$ to get $(x+2)(x-1)(x+1)$, the LCD.

Because $\frac{x+1}{x+1}=1$ and $\frac{x+2}{x+2}=1$, we are not changing the value of either rational expression, only its appearance.

Now we are ready to perform the indicated addition.

$$\frac{x+3}{x^2+x-2}+\frac{2}{x^2-1}$$ This is the given problem.

$$=\frac{x+3}{(x+2)(x-1)}+\frac{2}{(x+1)(x-1)}$$ Factor the denominators. The LCD is $(x+2)(x-1)(x+1)$.

$$=\frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)}+\frac{2(x+2)}{(x+2)(x-1)(x+1)}$$ Rewrite equivalent expressions with the LCD.

Step 3 Add numerators, putting this sum over the LCD.

$$=\frac{(x+3)(x+1)+2(x+2)}{(x+2)(x-1)(x+1)}$$

$$=\frac{x^2+4x+3+2x+4}{(x+2)(x-1)(x+1)}$$ Perform the multiplications in the numerator.

$$=\frac{x^2+6x+7}{(x+2)(x-1)(x+1)},\ x\neq-2,\ x\neq1,\ x\neq-1$$ Combine like terms in the numerator: $4x+2x=6x$ and $3+4=7$.

Step 4 If necessary, simplify. Because the numerator is prime, no further simplification is possible.

Check Point 8 Subtract: $\frac{x}{x^2-10x+25}-\frac{x-4}{2x-10}$.

6 Simplify complex rational expressions.

Complex Rational Expressions

Complex rational expressions, also called **complex fractions**, have numerators or denominators containing one or more rational expressions. Here are two examples of such expressions:

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$ Separate rational expressions occur in the numerator and the denominator.

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$ Separate rational expressions occur in the numerator.

One method for simplifying a complex rational expression is to combine its numerator into a single expression and combine its denominator into a single expression. Then perform the division by inverting the denominator and multiplying.

EXAMPLE 9 **Simplifying a Complex Rational Expression**

Simplify: $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x}}$.

Solution

Step 1 Add to get a single rational expression in the numerator.

$$1 + \frac{1}{x} = \frac{1}{1} + \frac{1}{x} = \frac{1 \cdot x}{1 \cdot x} + \frac{1}{x} = \frac{x}{x} + \frac{1}{x} = \frac{x+1}{x}$$

The LCD is $1 \cdot x$, or x.

Step 2 Subtract to get a single rational expression in the denominator.

$$1 - \frac{1}{x} = \frac{1}{1} - \frac{1}{x} = \frac{1 \cdot x}{1 \cdot x} - \frac{1}{x} = \frac{x}{x} - \frac{1}{x} = \frac{x-1}{x}$$

The LCD is $1 \cdot x$, or x.

Step 3 Perform the division indicated by the main fraction bar: Invert and multiply. If possible, simplify.

$$\frac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x}} = \frac{\dfrac{x+1}{x}}{\dfrac{x-1}{x}} = \frac{x+1}{x} \cdot \frac{x}{x-1} = \frac{x+1}{\cancel{x}_{1}} \cdot \frac{\cancel{x}^{1}}{x-1} = \frac{x+1}{x-1}$$

Invert and multiply.

Check Point 9 Simplify: $\dfrac{\dfrac{1}{x} - \dfrac{3}{2}}{\dfrac{1}{x} + \dfrac{3}{4}}$.

A second method for simplifying a complex rational expression is to find the least common denominator of all the rational expressions in its numerator and denominator. Then multiply each term in its numerator and denominator by this least common denominator. Because we are multiplying by a form of 1, we will obtain an equivalent expression that does not contain fractions in its numerator or denominator. Here we use this method to simplify the complex rational expression in Example 9.

$$\frac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x}} = \frac{\left(1 + \dfrac{1}{x}\right)}{\left(1 - \dfrac{1}{x}\right)} \cdot \frac{x}{x}$$

The least common denominator of all the rational expressions is x. Multiply the numerator and denominator by x. Because $\frac{x}{x} = 1$, we are not changing the complex fraction ($x \neq 0$).

$$= \frac{1 \cdot x + \dfrac{1}{x} \cdot x}{1 \cdot x - \dfrac{1}{x} \cdot x}$$

Use the distributive property. Be sure to distribute x to every term.

$$= \frac{x+1}{x-1}, \; x \neq 0, x \neq 1$$

Multiply. The complex rational expression is now simplified.

EXAMPLE 10 **Simplifying a Complex Rational Expression**

Simplify: $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$.

Solution We will use the method of multiplying each of the three terms, $\frac{1}{x+h}$, $\frac{1}{x}$, and h, by the least common denominator. The least common denominator is $x(x+h)$.

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\frac{\left(\frac{1}{x+h}-\frac{1}{x}\right)x(x+h)}{hx(x+h)}$$

Multiply the numerator and denominator by $x(x+h)$, $h \neq 0$, $x \neq 0$, $x \neq -h$.

$$=\frac{\frac{1}{x+h}\cdot x(x+h)-\frac{1}{x}\cdot x(x+h)}{hx(x+h)}$$

Use the distributive property in the numerator.

$$=\frac{x-(x+h)}{hx(x+h)}$$

Simplify: $\frac{1}{\cancel{x+h}}x(\cancel{x+h}) = x$ and $\frac{1}{\cancel{x}}\cdot\cancel{x}(x+h) = x+h$.

$$=\frac{x-x-h}{hx(x+h)}$$

Subtract in the numerator. Remove parentheses and change the sign of each term in parentheses.

$$=\frac{-\overset{1}{\cancel{h}}}{\underset{1}{\cancel{h}}x(x+h)}$$

Simplify: $x - x - h = -h$.

$$=-\frac{1}{x(x+h)},\ h \neq 0, x \neq 0, x \neq -h$$

Divide the numerator and denominator by h.

Check Point 10 Simplify: $\dfrac{\dfrac{1}{x+7}-\dfrac{1}{x}}{7}$.

Exercise Set P.6

Practice Exercises

In Exercises 1–6, find all numbers that must be excluded from the domain of each rational expression.

1. $\dfrac{7}{x-3}$
2. $\dfrac{13}{x+9}$
3. $\dfrac{x+5}{x^2-25}$
4. $\dfrac{x+7}{x^2-49}$
5. $\dfrac{x-1}{x^2+11x+10}$
6. $\dfrac{x-3}{x^2+4x-45}$

In Exercises 7–14, simplify each rational expression. Find all numbers that must be excluded from the domain of the simplified rational expression.

7. $\dfrac{3x-9}{x^2-6x+9}$
8. $\dfrac{4x-8}{x^2-4x+4}$
9. $\dfrac{x^2-12x+36}{4x-24}$
10. $\dfrac{x^2-8x+16}{3x-12}$
11. $\dfrac{y^2+7y-18}{y^2-3y+2}$
12. $\dfrac{y^2-4y-5}{y^2+5y+4}$